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Global Moduli of Equisingular Complete Curves

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INTRODUCTION

In this paper we fix an algebraically closed field k of arbitrary characteristic. By a curve we mean a proper irreducible reduced finite type k-scheme of dimension one. The genus of a curve means the genus of its nonsingular model.

DEFINITION 1. We say a unibranch singular point p of a curve C is of type d, if $\dim_k \bar{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p} = d$, where $\bar{\mathcal{O}}_{C,p}$ is the normalization of $\mathcal{O}_{C,p}$.

The purpose of this paper is to construct the global moduli space of equisingular curves of given genus whose singularities are all unibranch. Here "equisingular" means fixing the following data:

(1) number r of singular points,

(2) set of types $d_1, d_2, \dots, d_r$ of singular points.

As a matter of fact, we will prove the following theorem in Sections 3 and 4.

THEOREM. *For an integer $g \geqslant 2$ and an ordered set of positive integers $(d_1, d_2, \dots, d_r)$, define the contravariant functor $\mathscr{M}_{g,d_1,\dots,d_r}$ from the category of reduced Noetherian k-schemes to the category of sets as follows. For a reduced Noetherian k-scheme S, let $\mathscr{M}_{g,d_1,\dots,d_r}(S)$ be the set of isomorphism classes of diagrams*

$$\begin{tikzcd} \bar{\Gamma} \arrow[d, "\gamma"'] \arrow[dr, "\bar{\pi}"] & \\ \Gamma \arrow[r, "\pi"] & S \arrow[ul, bend right, "\bar{\sigma}_i"'] \arrow[l, bend left, "\sigma_i"] \end{tikzcd} \qquad (i = 1, 2, \dots, r)$$

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over S, *where*

(i) $\bar{\pi}: \bar{\Gamma} \to S$ *is a smooth family of curves of genus* g,

(ii) $\pi: \Gamma \to S$ *is a flat family of curves*,

(iii) $\pi \circ \gamma = \bar{\pi}$,

(iv) $\sigma_i, \bar{\sigma}_i$ $(i = 1, 2, \ldots, r)$ *are disjoint sections of* π, $\bar{\pi}$, *respectively, and* $\sigma_i = \gamma \circ \bar{\sigma}_i$ *for any* i,

(v) *for every point* s *of* S, *the singular locus of* Γ_s *is the set* $\{\sigma_1(s), \ldots, \sigma_r(s)\}$ *where* $\sigma_i(s)$ *is a unibranch singular point of type* d_i *for any* $i = 1, 2, \ldots, r$, *and* $\gamma_s: \bar{\Gamma}_s \to \Gamma_s$ *is the resolution of singularities whose restriction* $\bar{\Gamma}_s - \bigcup \bar{\sigma}_i(s) \to \Gamma_s - \bigcup \sigma_i(s)$ *is an isomorphism.*

Then there exists a coarse moduli for $\mathscr{M}_{g,d_1,\ldots,d_r}$ *and the moduli space is a connected reduced quasi-projective* k*-scheme.*

Here we recall the terminology "coarse moduli." A k-scheme M and a morphism η from the functor $\mathscr{M}_{g,d_1,\ldots,d_r}$ to the functor $h_M(*) = \mathrm{Hom}(*, M)$ represented by M is called a coarse moduli for $\mathscr{M}_{g,d_1,\ldots,d_r}$ if

(1) $\eta(\mathrm{Spec}\, k): \mathscr{M}_{g,d_1,\ldots,d_r}(\mathrm{Spec}\, k) \to h_M(\mathrm{Spec}\, k)$ is an isomorphism,

(2) given a reduced Noetherian k-scheme N and any morphism ψ from $\mathscr{M}_{g,d_1,\ldots,d_r}$ to the representable functor h_N, there is the unique morphism $\chi: h_M \to h_N$ such that $\psi = \chi \circ \eta$.

1. Moduli of Subrings of a Normally Flat Algebra

In our earlier paper [3], we get the fine moduli space of all subrings of fixed colength of a local ring and see the connectedness of the space. In this section we develop its relative version which will be used in the proof of the theorem.

Definition 2. For a Noetherian k-scheme S, we call the diagram $\mathscr{O}_S \overset{\pi^*}{\underset{\sigma^*}{\rightleftarrows}} \bar{\mathscr{O}}$ of quasi-coherent $\mathscr{O}_S$-algebras satisfying (i), (ii), (iii) a normally flat deformation of local rings on S.

(i) $\sigma^* \circ \pi^* = \mathrm{id}_{\mathscr{O}_S}$,

(ii) denote $\mathrm{Ker}\, \sigma^*$ by $\mathscr{I}$, then $\mathrm{gr}_{\mathscr{I}}(\bar{\mathscr{O}})$ is a flat $\mathscr{O}_S$-module, and

(iii) for any point s of S, $\bar{\mathscr{O}} \otimes_{\mathscr{O}_S} k(s)$ is a local ring with the maximal ideal $\mathscr{I} \otimes_{\mathscr{O}_S} k(s)$.

Definition 3. Fix a normally flat deformation of local rings $\mathscr{O}_S \overset{\pi^*}{\underset{\sigma^*}{\rightleftarrows}} \bar{\mathscr{O}}$ and integers $d > 0$, $s > 0$. For a Noetherian S-scheme S', define $\mathscr{F}^{d,s}_{\bar{\mathscr{O}}/S}(S')$ to be the set of subalgebras $\mathscr{O}$ of $\bar{\mathscr{O}} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}$ which contain $\mathscr{I}^s \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}$ with

$\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}/\mathcal{O}$ being locally free $\mathcal{O}_{S'}$-modules of rank d. And for an S-morphism $f: S' \to S''$, define $\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}(f)$ to be the map which sends an element of $\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}(S'')$ to the pull-back sheaf of it in $\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}(S')$.

Note that $\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}$ is a contravariant functor from the category of Noetherian S-schemes to the category of sets.

PROPOSITION 1. *$\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}$ is representable by a projective scheme over S.*

Proof. There is the minimum element in the set {closed subscheme Z of $\mathrm{Grass}^d(\bar{\mathcal{O}}/\mathscr{I}^s)$, where $\mathcal{O} \otimes_{\mathcal{O}_{\mathrm{Grass}}} \mathcal{O}_Z$ is an $\mathcal{O}_{\underline{Z}}$-subalgebra of $\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} \mathcal{O}_Z$}, where $\mathcal{O}$ is the universal subsheaf on $\mathrm{Grass}^d(\bar{\mathcal{O}}/\mathscr{I}^s)$ (cf. Lemma 2 in [3]). This minimum element is our representing scheme.

DEFINITION 4. The representing scheme of Proposition 1 is denoted by $\mathrm{Ter}^{d,s}(\bar{\mathcal{O}}/S)$.

PROPOSITION 2. *The reduced structure of* $\mathrm{Ter}^{d,s}(\bar{\mathcal{O}}/S)$ *does not depend on s, if $s \geqslant 2d$.*

Proof. Since $\mathscr{F}^{d,2d}_{\bar{\mathcal{O}}/S}$ is a subfunctor of $\mathscr{F}^{d,s}_{\bar{\mathcal{O}}/S}$ for $s \geqslant 2d$, there is the canonical immersion i: $\mathrm{Ter}^{d,2d}(\bar{\mathcal{O}}/S) \to \mathrm{Ter}^{d,s}(\bar{\mathcal{O}}/S)$. If we restrict i on a fibre of a closed point s of S, we have a bijective immersion $\mathrm{Ter}^{d,2d}(\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} k(s)) \to \mathrm{Ter}^{d,s}(\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} k(s))$ by Proposition 2 of [3]. Therefore i is a bijective immersion.

Remark 1. For S-scheme S', let $\mathscr{F}^{d}_{\bar{\mathcal{O}}/S}$ be the set of all subalgebras $\mathcal{O}$ of $\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ whose quotient $\bar{\mathcal{O}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}/\mathcal{O}$ are locally free $\mathcal{O}_{S'}$-modules of rank d. Then $\mathscr{F}^{d}_{\bar{\mathcal{O}}/S}$ is a contravariant functor from the category of S-schemes to the category of sets. The reduced scheme of Proposition 2, which we will denote by $\mathrm{ter}^d(\bar{\mathcal{O}}/S)$, is the representing scheme of the functor which is the restriction of $\mathscr{F}^{d}_{\bar{\mathcal{O}}/S}$ to the subcategory consisting of all reduced Noetherian S-schemes.

Remark 2. Especially if $\mathcal{O}_S \underset{\sigma^*}{\overset{\pi^*}{\rightleftarrows}} \bar{\mathcal{O}}$ is a normally flat deformation of descrete valuation rings and if S is reduced, then $\mathrm{ter}^d(\bar{\mathcal{O}}/S)$ is a locally trivial fibre space on S with the fibre $\mathrm{ter}^d(k[[x]])$. In fact, let $\{U_i\}$ be an open covering such that $\bar{\mathcal{O}}/\mathscr{I}2d|_{U_i}$ is a free $\mathcal{O}_S$-module, then

$$\begin{aligned}\mathrm{ter}^d((\bar{\mathcal{O}}\,|\,U_i)/U_i) = \mathrm{ter}^d((\bar{\mathcal{O}}/\mathscr{I}^{2d})/U_i) &= \mathrm{ter}^d((k[[x]]/(x^{2d})) \otimes_k \mathcal{O}_{U_i}) \\ &= \mathrm{ter}^d(k[[x]]) \times U_i.\end{aligned}$$

2. A Certain Functorial Property of Stability

Let G be a geometrically reductive algebraic group over k acting on a k-scheme X. We say a closed point x of X to be properly stable by the action of G if (i) there is a G-invariant affine open neighbourhood U of x, (ii) for any closed point y of U the morphism $\mu_y: G \to U$ defined by $\mu_y(g) = \mu(g, y)$ is proper, where $\mu: G \times X \to X$ is the action of G on X. When all points of X are properly stable, X is called a properly G-stable variety. By [4], a properly G-stable variety has the geometric quotient by G.

Assume G acts on two varieties Y, X and $f: Y \to X$ is a G-linear morphism. Consider the problem "when the proper stability of X can be lifted to Y." If f is an affine morphism, this problem is trivially true. In this section we consider the case f is a projective morphism, which will be effective in Section 3.

Proposition 3. *Let P and X be two k-schemes and assume that G acts on them. Let $f: P \to X$ be a projective G-linear morphism. Assume X is properly stable and there exists an f-ample G-linearized invertible sheaf on P. Then P is also properly stable. If X/G is quasi-projective, so is P/G.*

Proof. We may suppose that X is affine and $\mu_x: G \to X$ is proper for any closed point $x \in X$. Let $\mu: G \times X \to X$, $\mu': G \times P \to P$ be actions of G on X, P respectively. If we put $S = \Gamma(G, \mathcal{O}_G)$, then by the existence of an f-ample G-linearized invertible sheaf, there is a graded $\mathcal{O}_X$-algebra $\mathscr{A}$ and a grade preserving homomorphism $\mu'^*: \mathscr{A} \to S \otimes_k \mathscr{A}$ such that $P = \operatorname{Proj} \mathscr{A}$ and μ'^* gives the action of G on P. If we put $V = \operatorname{Spec} \mathscr{A}$, then μ'^* also gives an action of G on V. Denote the zero section of V by Z. Then Z is an invariant closed subscheme of V and the canonical projections $\pi: V - Z \to P$ and $g: V \to X$ are G-linear. For any closed point p of P, $\mu_{f(p)} = f \circ \mu'_p$ is proper. So by [2, II, 5.4.3], μ'_p is also proper, and a fortiori any orbit $O(p)$ is closed in P. Similarly, any orbit in V is closed. Now we have only to show the existence of an invariant affine open neighbourhood of each point p of P. For any point p of P, take a point p^* of V lying over p. Then $Z \cap O(p^*) = \phi$, and so there is an invariant function F^* on V such that $F^* = 0$ on Z and $F^* \neq 0$ on $O(p^*)$ (cf. [5, Lemma 3.3]). Since $\Gamma(V, \mathcal{O}_V) = \Gamma(X, \mathscr{A})$, we can write down as $F^* = F^*_{k_1} + F^*_{k_2} + \cdots + F^*_{k_r}$ where $F^*_{k_i}$ is a homogeneous element of $\Gamma(X, \mathscr{A})$ of degree k_i for each i. The equation $F^* = 0$ on Z implies that all k_i are positive. On the other hand there is at least one k_i satisfying $F^*_{k_i}(p^*) \neq 0$. Let F be the section of $\mathcal{O}_P(k_i)$ corresponding to such function. We now get an invariant affine open neighbourhood P_F of p. The second assertion is checked by using stability of X with respect to a G-linear invertible sheaf.

COROLLARY 1. *Let $f: P \to X$ be a blowing up with a G-invariant center. Assume X is properly stable. Then P is also properly stable.*

Proof. The $\mathscr{O}_P$-ideal of the exceptional divisor is an f-ample G-linearized invertible sheaf.

COROLLARY 2. *Let $f: P \to X$ be a G-linear projective bundle over a k-scheme X. Assume X is properly stable. Then P is also properly stable.*

Proof. Let n be the dimension of the fibre of f, then $(\bigwedge^n \Omega_{P/X})^{-1}$ is an f-ample G-linearized invertible sheaf.

3. CONSTRUCTION OF THE MODULI SPACE

In this section we will construct a coarse moduli scheme of the theorem noted in the Introduction. Before beginning the construction, we prepare some results contained in [1, 4].

PROPOSITION 4. *For any integer $v \geqslant 3$, $g \geqslant 2$, and $r \geqslant 1$, take an integer n to be $v(2g-2)-g$. Then the following functor $\mathscr{G}$ from the category of Noetherian k-schemes to the category of sets is representable by a quasi-projective k-scheme; For a Noetherian k-scheme S, define $\mathscr{G}(S)$ to be the sets of commutative diagrams*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "\pi"] & \mathbb{P}^n \times S \arrow[d, "p_2"] \\ & S \arrow[ul, bend left, "\sigma_i"] \end{tikzcd} \qquad (i = 1, \ldots, r)$$

where

(i) *$\pi: C \to S$ is a smooth family of curves of genus g,*

(ii) *the invertible sheaf on C induced by $\mathscr{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\Omega_{C/S} \otimes p_2^* \mathscr{L}$, for a suitable invertible sheaf $\mathscr{L}$ on S,*

(iii) *for every geometric point s of S, the fibre C_s of C over s spans the projective space $\mathbb{P}_k^n$,*

(iv) *$\sigma_1, \ldots, \sigma_r$ are sections of π and disjoint to each other.*

Let us denote the representing scheme by $\mathscr{H}_{g,r}$. Note that the action of $PGL(n)$ on $\mathbb{P}^n$ induces an action on $\mathscr{H}_{g,r}$.

PROPOSITION 5. *$\mathscr{H}_{g,r}$ is an irreducible reduced non-singular k-scheme. And there exists the geometric quotient $\mathscr{H}_{g,r}/PGL(n)$ which is quasi-projective and the coarse moduli scheme of non-singular curves of genus g with r-ordered points.*

Construction. For the simplification of notation we write $\mathscr{H}$ instead of $\mathscr{H}_{h,r}$. Let $\mathscr{U} \rightleftarrows^{\Pi}_{\Sigma_i} \mathscr{H}$ $(i = 1,..., r)$ be the universal family on $\mathscr{H}$. For each i we define a sheaf $\bar{\mathscr{O}}_i$ on $\mathscr{H}$ by $\Gamma(U, \bar{\mathscr{O}}_i) = \bigcap_{p \in U} \mathscr{O}_{\mathscr{U}, \Sigma_i(p)}$, where U is an open subset of $\mathscr{H}$. Note that this intersection is defined in the function field of $\mathscr{U}$ which is irreducible and reduced. Then $\bar{\mathscr{O}}_i$ is a quasi-coherent $\mathscr{O}_{\mathscr{H}}$-algebra and Π and Σ_i induce a normally flat deformation $\mathscr{O}_{\mathscr{H}} \rightleftarrows^{\bar{\pi}_i}_{\bar{\sigma}_i} \bar{\mathscr{O}}_i$ of discrete valuation rings. Moreover we can easily check that $(\bar{\mathscr{O}}_i)_p = \mathscr{O}_{\mathscr{U}, \Sigma_i(p)}$ for any $p \in \mathscr{H}$. Denote $\mathrm{ter}^{d_i}(\bar{\mathscr{O}}_i/\mathscr{H})$ by $\mathscr{E}_i$ and the fibre product $\mathscr{E}_1 \times_{\mathscr{H}} \cdots \times_{\mathscr{H}} \mathscr{E}_r$ by $\mathscr{E}$. In the following we will introduce an action of $PGL(n)$ on $\mathscr{E}$ which is compatible with the action on $\mathscr{H}$. Let $\mu: PGL(n) \times \mathscr{H} \to \mathscr{H}$ define the action of $PGL(n)$ on $\mathscr{H}$, then

$$\mu^* \bar{\mathscr{O}}_i = p_2^* \bar{\mathscr{O}}_i \qquad \text{for } i = 1,..., r, \tag{1}$$

where $p_2: PGL(n) \times \mathscr{H} \to \mathscr{H}$ is the projection to the second factor. Because $PGL(n) \times \mathscr{U} \simeq (PGL(n) \times \mathscr{H}) \times_\mu \mathscr{U}$ and $PGL(n) \times \Sigma_i(\mathscr{H}) \simeq (PGL(n) \times \mathscr{H}) \times_\mu \Sigma_i(\mathscr{H})$.

Let $\psi_i: \mathscr{E}_i \to \mathscr{H}$ be the structural morphism, $\bar{p}_2: PGL(n) \times \mathscr{E}_i \to \mathscr{E}_i$ be the projection, and $\mathscr{O}_i \subset \bar{\mathscr{O}}_i \otimes_{\mathscr{O}_{\mathscr{H}}} \mathscr{O}_{\mathscr{E}_i}$ be the universal subalgebra on $\mathscr{E}_i$. Then we get that $\bar{p}_2^*(\bar{\mathscr{O}}_i \otimes \mathscr{O}_{\mathscr{E}_i}) = \bar{p}_2^* \psi_i^* \bar{\mathscr{O}}_i = (1 \times \psi_i)^* p_2^* \bar{\mathscr{O}}_i$, where the last term is isomorphic to $(1 \times \psi_i)^* \mu^* \bar{\mathscr{O}}_i$ by Eq. (1). So the subring $\bar{p}_2^* \mathscr{O}_i$ of $\bar{p}_2^*(\bar{\mathscr{O}}_i \otimes \mathscr{O}_{\mathscr{E}_i})$ gives a subring of $(1 \times \psi_i)^* \mu^* \bar{\mathscr{O}}_i$ which is an element of $\mathscr{F}^{d_i}_{\bar{\mathscr{O}}_i/\mathscr{H}}(PGL(n) \times \mathscr{E}_i)$, where $PGL(n) \times \mathscr{E}_i$ is considered as $\mathscr{H}$-scheme by the morphism $\mu(1 \times \psi_i)$. Since $\mathscr{F}^{d_i}_{\bar{\mathscr{O}}_i/\mathscr{H}}$ is representable by $\mathscr{E}_i$, this subring corresponds to an $\mathscr{H}$-morphism $\mu_i: PGL(n) \times \mathscr{E}_i \to \mathscr{E}_i$. Notice that $\psi_i \circ \mu_i = \mu(1 \times \psi_i)$, so μ_i is an action of $PGL(n)$ on $\mathscr{E}_i$ which is compatible with the one on $\mathscr{H}$. From $\{\mu_i\}$ we obtain an action $\bar{\mu}$ of $PGL(n)$ on $\mathscr{E}$ which is compatible with the one on $\mathscr{H}$. Our assertion is that the geometric quotient of $\mathscr{E}$ by $PGL(n)$ is our coarse moduli scheme.

PROPOSITION 6. *There exists the geometric quotient of* $\mathscr{E}$ *by* $PGL(n)$.

Proof. If there are the geometric quotients $\mathscr{E}_i/PGL(n)$ of $\mathscr{E}_i$ by $PGL(n)$ for $i = 1,..., r$, then by Proposition 5, $(\mathscr{E}_1/PGL(n)) \times_{\mathscr{H}/PGL(n)} \cdots \times_{\mathscr{H}/PGL(n)} (\mathscr{E}_r/PGL(n))$ must be the geometric quotient of $\mathscr{E}$. So we have only to show the existence of the geometric quotient of $\mathscr{E}_i$ by $PGL(n)$ for each i. By the Plücker embedding, $\mathscr{E}_i$ is considered as a closed subscheme of $\mathbb{P}(\bigwedge^d \bar{\mathscr{O}}_i/\mathscr{I}^{2d})$, where $\mathscr{I}$ is the kernel of $\sigma^*: \bar{\mathscr{O}}_i \to \mathscr{O}_{\mathscr{H}}$. The equation $p_2^*(\bigwedge^d \bar{\mathscr{O}}_i/\mathscr{I}\ 2d) = \mu^*(\bigwedge^d \bar{\mathscr{O}}_i/\mathscr{I}^{2d})$ induces an action on $\mathbb{P}(\bigwedge^d \bar{\mathscr{O}}_i/\mathscr{I}^{2d})$ which is compatible with the action on $\mathscr{E}_i$. By [4], it suffices to show that any closed point of $\mathbb{P}(\bigwedge^d \bar{\mathscr{O}}_i/\mathscr{I}^{2d})$ is properly stable. This follows immediately from Corollary 2.

4. Verification

In this section we will show that the geometric quotient $\mathscr{E}/PGL(n)$ is our required moduli scheme of the theorem.

Proposition 7. *There exists a natural morphism of functors* $\Phi\colon h_{\mathscr{E}} \to \mathscr{M}_{g,d_1,d_2,\ldots,d_r}$.

Proof. For the simplification of notation we prove the case $r=1$. The case $r>1$ will follow easily. We will drop the subscript 1. Definition of $\bar{\mathscr{O}}$ leads us to the canonical injection $\varphi\colon \operatorname{Spec}\bar{\mathscr{O}} \to \mathscr{U}$ satisfying that for any point p of $\operatorname{Spec}\bar{\mathscr{O}}$, φ induces an isomorphism between $\mathscr{O}_{\operatorname{Spec}\bar{\mathscr{O}},p}$ and $\mathscr{O}_{\mathscr{U},\varphi(p)}$. By the diagram $\mathscr{O}_{\mathscr{H}} \underset{\bar{\sigma}^*}{\overset{\bar{\pi}^*}{\rightleftarrows}} \bar{\mathscr{O}}$, we get the diagram $\mathscr{H} \underset{\bar{\sigma}}{\overset{\bar{\pi}}{\leftrightarrows}} \operatorname{Spec}\bar{\mathscr{O}}$ where $\bar{\sigma}$ is a section of $\bar{\pi}$ and equations $\Pi\cdot\varphi=\bar{\pi}$, $\Sigma=\varphi\circ\bar{\sigma}$ hold. To construct Φ, it suffices to determine the image of $\mathrm{id}_{\mathscr{E}} \in h_{\mathscr{E}}(\mathscr{E})$ in $\mathscr{M}_{g,d}(\mathscr{E})$. We have two kinds of families on $\mathscr{E}$; one is $\bar{\Gamma}=\mathscr{U}\times_{\mathscr{H}}\mathscr{E} \underset{\Sigma'}{\overset{\Pi'}{\rightleftarrows}} \mathscr{E}$ which is the pull-back family of $\mathscr{U} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathscr{H}$ by $\psi\colon \mathscr{E}\to\mathscr{H}$, and the other is the universal dominated family $\mathscr{O} \subset \bar{\mathscr{O}} \otimes \mathscr{O}_{\mathscr{E}}$. By the latter we obtain a family $\operatorname{Spec}\mathscr{O}$ on $\mathscr{E}$ such that the diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathscr{O} \arrow[drr] & & \operatorname{Spec}\bar{\mathscr{O}}\times_{\mathscr{H}}\mathscr{E} \arrow[ll, "h"'] \arrow[d, shift left, "\bar{\pi}\times 1_{\mathscr{E}}"] \\
& & \mathscr{E} \arrow[u, shift left, "\bar{\sigma}-1_{\mathscr{E}}"]
\end{tikzcd}$$

restriction h' of h on $(\operatorname{Spec}\bar{\mathscr{O}})\times\mathscr{E}-(\bar{\sigma}\times 1_{\mathscr{E}})(\mathscr{E})$ is an isomorphism to $(\operatorname{Spec}\mathscr{O})-\sigma(\mathscr{E})$. Recall that there is the canonical injection $\varphi'\colon(\operatorname{Spec}\bar{\mathscr{O}})-\bar{\sigma}(\mathscr{H}) \hookrightarrow \mathscr{U}-\Sigma(\mathscr{H})$ which is the restriction of φ, then we have the canonical injection $\varphi'\times 1_{\mathscr{E}}\colon(\operatorname{Spec}\bar{\mathscr{O}})\times\mathscr{E}-(\bar{\sigma}\times 1_{\mathscr{E}})(\mathscr{E})\to \bar{\Gamma}-\Sigma'(\mathscr{E})$. Patching $\bar{\Gamma}-\Sigma'(\mathscr{E})$ and $\operatorname{Spec}\mathscr{O}$ by $(\varphi'\times 1_{\mathscr{E}})\circ h'^{-1}$ we get a local ringed space $(\Gamma,\mathscr{O}_\Gamma)$. We claim that $(\Gamma,\mathscr{O}_\Gamma)$ is a k-scheme which is of finite type over $\mathscr{E}$. By the construction of Γ, there is a map $\gamma\colon\bar{\Gamma}\to\Gamma$ of ringed spaces which is an isomorphism on the outside of the section and the induced homomorphism $\gamma^*\colon\mathscr{O}_\Gamma\to\mathscr{O}_{\bar{\Gamma}}$ is the canonical inclusion. Let $\bar{U}$ be an affine open subset of $\bar{\Gamma}$ and U be the open subset of Γ corresponding to $\bar{U}$. Denote the rings of sections $\Gamma(\bar{U},\mathscr{O}_{\bar{\Gamma}})$, $\Gamma(U,\mathscr{O}_\Gamma)$, $\Gamma(\Pi'(\bar{U}),\mathscr{O}_{\mathscr{E}})$ by $\bar{R}$, R, S respectively. Note that $\Pi'(\bar{U})$ is open in $\mathscr{E}$, since Π' is smooth and so an open map. We will show that R is of finite type over S and $U\simeq\operatorname{Spec}R$. We may assume that $\bar{U}$ intersects the section $\Sigma'(\Pi'(\bar{U}))$. Denote $\Gamma(\bar{U},\operatorname{Ker}\Sigma'^*)$ by I, then $I^{2d}\subset R$ by the proof of Proposition 2 of [3]. Then $\bar{R}$ is integral over R, because $\bar{R}\simeq I\oplus S$ by the diagram $S \underset{\Sigma'^*}{\overset{\Pi'^*}{\rightleftarrows}} \bar{R}$. Since $\bar{R}$ is of finite type over S, it follows that R is of finite type by [6, Chap. III, n^0 12, lemme 10].

Next, for a point $\bar{x}$ of $\bar{U}$, let $\mathscr{P}$ be a prime ideal in $\bar{R}$ defining $\bar{x}$ and $x \in U$ be the image of $\bar{x}$ by γ. In order to show that $U \simeq \operatorname{Spec} R$, it suffices to show that $\mathscr{O}_{U,x} = R_{\mathscr{P} \cap R}$. We may assume that $\bar{x}$ is on the section. It is clear that the right-hand side is contained in the left. For the converse, if we take an element $f \in \mathscr{O}_{U,x}$, then f is represented by c/d, where $c, d \in \bar{R}$ and d is a unit of $\bar{R}$. The image of d in $\bar{R}_{\mathscr{P}}/I_{\mathscr{P}}$ is also a unit. Noticing that $\bar{R}/I = S$ and $\bar{R}_{\mathscr{P}}/I_{\mathscr{P}} = \mathscr{O}_{\mathscr{E}, \Pi'(\bar{x})}$, we see that there exists a function $g \in S$ such that the images of d in $S_g = \bar{R}_g/I_g$ and in $\bar{R}_g/I_g^{2d}$ are units and that $f \in \bar{R}_g$. Then there is a function $h \in \bar{R}$ and an integer $s > 0$ so that $(h/g^s)d \equiv 1$ modulo I_g^{2d}. Since $I_g^{2d} \subset R_g$ and $g \in S$, hd is contained in $R_g \cap \bar{R}$. Now $hc = fhd$ is contained in $R_g \cap \bar{R}$. Hence $f = hc/hd$ is contained in $(R_g \cap \bar{R})_{\mathscr{P} \cap R_g \cap \bar{R}}$ so a fortiori in $R_{\mathscr{P} \cap R}$. Consequently we get a k-scheme Γ of finite type over $\mathscr{E}$ and a commutative diagram

$$\begin{array}{ccc} \bar{\Gamma} & & \\ {\scriptstyle\gamma}\downarrow & \searrow \nwarrow^{\Sigma'} & \\ \Gamma & \xrightarrow{\pi'} & \mathscr{E}. \end{array}$$

If we set $\sigma = \gamma \circ \Sigma'$, then it is easy to verify that this diagram is an element of $\mathscr{M}_{g,d}(\mathscr{E})$. If we let $\mathrm{id}_{\mathscr{E}}$ correspond to this element, then this completes the proof of Proposition 7.

Henceforth we will omit the subscripts g, $d_1, d_2, \ldots, d_r$ of $\mathscr{M}_{g,d_1,\ldots,d_r}$. The action $\bar{\mu}: PGL(n) \times \mathscr{E} \to \mathscr{E}$ induces an action $\hat{\mu}: h_{PGL(n)} \times h_{\mathscr{E}} \to h_{\mathscr{E}}$ of functors. Define $\mathscr{M}'(S)$ to be the quotient $h_{\mathscr{E}}(S)/h_{PGL(n)}(S)$ for a reduced Noetherian k-scheme S. Then $\mathscr{M}'$ is a contravariant functor and Φ factors as follows; $h_{\mathscr{E}} \to^{\Phi'} \mathscr{M}' \to^{\iota} \mathscr{M}$.

Proposition 8. (i) *ι is injective*, (ii) *for any $\alpha \in \mathscr{M}(s)$, there exists an open covering $\{U_i\}$ of S and a collection $\{\alpha_i\}$ of elements $\alpha_i \in \mathscr{M}'(U_i)$ such that $\alpha|_{U_i} = \iota(\alpha_i)$ for any i.*

Proof. (i) For $\alpha, \beta \in h_{\mathscr{E}}(S)$, assume $\Phi(\alpha) = \Phi(\beta)$ in $\mathscr{M}(S)$. Then α, β induce families of non-singular curves of genus g on S;

$$\begin{array}{ccc} \bar{\Gamma} & \hookrightarrow & \mathbb{P}^n \times S \\ & \searrow & \downarrow \\ & & S \end{array}, \qquad \begin{array}{ccc} \bar{\Gamma}' & \hookrightarrow & \mathbb{P}^n \times S \\ & \searrow & \downarrow \\ & & S \end{array}$$

respectively, and subalgebras $\mathscr{O}_i \subset \alpha^* \psi^* \bar{\mathscr{O}}_i$, $\mathscr{O}'_i \subset \beta^* \psi^* \bar{\mathscr{O}}_i$ of colength d_i for each $i = 1, 2, \ldots, r$, respectively. By the assumption there exist isomorphisms $g: \bar{\Gamma} \to \bar{\Gamma}'$ and $g_i^*: \beta^*\psi^*\bar{\mathcal{O}}_i \simeq \alpha^*\psi^*\bar{\mathcal{O}}_i$ which corresponds a subring $\mathcal{O}_i'$ to a subring $\mathcal{O}_i$ for each i. By Proposition 5.2 of [4], the S-isomorphism g can be extended to an automorphism of $\mathbb{P}^n \times S$, and this shows that $\Phi'(\alpha) = \Phi'(\beta)$ in $\mathscr{M}'(S)$.

(ii) An element $\alpha \in \mathscr{M}(S)$ is an isomorphism class of a diagram

$$\begin{tikzcd} \bar{\Gamma} \arrow[d, "\gamma"'] \arrow[dr, "\bar{\pi}"'] & \\ \Gamma \arrow[r, shift left, "\pi"] & S \arrow[l, shift left, "\sigma_j"] \arrow[ul, shift right, "\bar{\sigma}_j"'] \end{tikzcd} \qquad (j = 1, 2, \ldots, r) \qquad \text{over } S.$$

By Proposition 5.2 of [4] there exists a covering $\{U_i\}$ of S such that the family $\bar{\Gamma}|_{U_i} \overset{\bar{\pi}}{\underset{\bar{\sigma}_j}{\rightleftarrows}} U_i$ comes from $h_{\mathscr{X}}(U_i)$ and $\Gamma|_{U_i} \overset{\pi}{\underset{\sigma_j}{\rightleftarrows}} U_i$ defines a dominated family $\mathcal{O}_j \subset \bar{\mathcal{O}}_j \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_{U_i}$ for each $j = 1, 2, \ldots, r$. These dominated families on U_i determine an element of $h_{\mathscr{E}}(U_i)$. Q.E.D.

By the above propositions we get the canonical bijection of sets, $\mathrm{Hom}(\mathscr{M}, h_N) \simeq \{f \in \mathrm{Hom}(\mathscr{E}, N) \mid f \circ p_2 = f \circ \bar{\mu}\}$, for any reduced Noetherian k-scheme N. If we put $N = \mathscr{E}/PGL(n)$ in the above bijection, then we get a natural morphism of functors $\eta: \mathscr{M} \to h_{\mathscr{E}/PGL(n)}$ corresponding to the natural surjection $\omega: \mathscr{E} \to \mathscr{E}/PGL(n)$. We can check that $(\mathscr{E}/PGL(n), \eta)$ is the coarse moduli for $\mathscr{M}$ by the same argument as Proposition 5.4 of [4].

PROPOSITION 9. *$\mathscr{E}/PGL(n)$ is reduced, connected and quasi-projective over k.*

Proof. Since $\mathrm{ter}^{d_i}(\bar{\mathcal{O}}_i/\mathscr{X})$ is a locally trivial fibre space with the fibre $\mathrm{ter}^{d_i}(k[[x]])$ on $\mathscr{X}$, the proposition follows.

Remark 3. We already have coarse moduli schemes not only of non-singular curves of given genus but also of polarized Abelian varieties and of polarized $K3$-surfaces by means of algebraic construction. The moduli scheme of unibranch equisingular such objects can be constructed by the same way as this articles.

REFERENCES

1. P. DELIGNE AND D. MUMFORD, The irreducibility of the space of curves of given genus, *Publ. I.H.E.S.* **36** (1969), 75–109.
2. A. GROTHENDIECK AND J. DIEUDONNÉ, Éléments de géométrie algébrique, II. *Publ. I.H.E.S.* **8** (1961).

3. S. Ishii, Moduli of subrings of a local ring, *J. Algebra* **67** (1980), 504–516.
4. D. Mumford, "Geometric Invariant Theory," Springer-Verlag, Berlin/Heidelberg/New York, 1965.
5. P. E. Newstead, "Introduction to Moduli Problems and Orbit Spaces," Tata Lecture Note 51, Springer-Verlag, Berlin/Heidelberg/New York, 1978.
6. J. P. Serre, "Groupes algébriques et corps de classes," Herman, Paris, 1959.